Summary of Woit's New Approach to Path Integrals

Grok, based on Peter Woit's Paper

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Abstract

In his paper *Topological Quantum Theories and Representation Theory*, Peter Woit proposes a novel approach to path integrals for quantizing a simple quantum system with a group representation Hilbert space. Addressing issues in coherent state path integrals, such as ill-defined normalization and incorrect tracing, Woit employs a supersymmetric quantum mechanics (SSQM) path integral to compute the index of a Dirac operator twisted by a line bundle. This approach leverages index theory and geometric quantization, ensuring only the desired representation contributes to the partition function. The method yields the dimension and character of representations via localization formulas and extends to loop groups and Chern-Simons theory.

1 Introduction

Peter Woit's paper explores a quantum system where the Hilbert space is a unitary representation V_R of a compact Lie group G, with a classical phase space as the flag manifold G/T. Traditional coherent state path integrals over G/T suffer from issues like violation of the Heisenberg uncertainty principle and incorrect tracing over non-holomorphic sections. Woit introduces a supersymmetric quantum mechanics (SSQM) path integral to resolve these problems, aligning quantization with integration in equivariant K-theory.

2 Coherent State Path Integrals and Their Limitations

For a quantum system with Hilbert space V_R, the partition function is the character $Z = \mathrm{Tr}_R(U(T))$, where $U(T) \in G$. Woit attempts to compute this via a coherent state path integral over the phase space $M = G/T$, using the resolution of the identity:

$$1 = \frac{1}{\Gamma'} \int_M |z\rangle\langle z|,$$

where $|z\rangle$ are coherent states, and the integral uses the symplectic measure on M. The inner product $\langle g_1 \cdot z_0 | g_0 \cdot z_0 \rangle$ is expressed as:

$$K(g_1, g_0) = \frac{1}{\Gamma''} \int_{g_0 \to g_1} e^{\int_0^1 d\tau(\omega + i\theta(\dot{g}))},$$

with $\omega = \frac{d\theta}{2\pi}$ the symplectic 2-form and θ the connection 1-form of the line bundle L_{λ_V}.

However, this approach faces issues:

- **Continuity Assumption**: Assuming continuous paths violates the Heisenberg uncertainty principle, as conjugate variables are specified simultaneously.
- **Normalization**: Without a damping term like $e^{-\frac{1}{2}\int |\dot{x}|^2}$, the integral's normalization is ill-defined.
- **Incorrect Trace**: The trace $\mathrm{Tr}_V(1) = \dim V$ includes contributions from all sections of $\Gamma(L_{\lambda_V})$, not just the holomorphic sections defining V.

3 Supersymmetric Quantum Mechanics Path Integral

To address these issues, Woit proposes an SSQM path integral to compute the index of the Dirac operator on G/T twisted by L_{λ_V}, which corresponds to the representation V. The index is expressed over the loop space ΩM:

$$\text{index } D = \int_{\Omega M} e^{\frac{1}{2}(d-i_X)(\alpha+\alpha_1)},$$

where:

- X is the vector field generating loop rotations.
- α is the 1-form dual to X.
- $\alpha_1 = 2i \oint d\tau A(x(\tau))$, with A the connection on L_{λ_V}.

The Lagrangian is:

$$L = \int d\tau \frac{1}{2} \left(\left| \frac{dx}{d\tau} \right|^2 - \psi^i D_\tau \psi^i \right),$$

and the partition function is:

$$Z = \int [dx][d\psi] e^{-L}.$$

3.1 Key Features

- **Damping Term**: The term $e^{-\frac{1}{2}\int |\dot{x}|^2}$ suppresses discontinuous paths, resolving normalization issues.
- **Fermionic Variables**: The fermionic fields ψ^i cancel contributions from non-zero modes of the Dirac operator, ensuring only holomorphic sections (i.e., $V = \Gamma_{\text{hol}}(L_{\lambda_V})$) contribute.
- **Localization**: The Atiyah-Bott localization formula reduces the integral to the fixed point set M:

  $$\int_{\Omega M} \mu = \int_M \frac{\mu|_M}{\chi(X, N)},$$

  where $\chi(X, N)$ is the equivariant Euler characteristic, yielding:

  $$\text{index } D = \hat{A}(M) \wedge \text{ch}(L_{\lambda_V})[M].$$

4 Representation Dimension and Character

The SSQM path integral computes the dimension of the representation:

$$\dim V = \frac{\int_{O(\lambda_V+\delta)} e^{\omega}}{\int_{O(\delta)} e^{\omega'}},$$

where $O(\lambda_V + \delta)$ and $O(\delta)$ are flag manifolds, δ is half the sum of positive roots, and ω, ω' are symplectic forms.

For the character, the vector field X is modified to include the action of a group element g, giving:

$$\text{tr}(e^X) = \frac{\int_{O(\lambda_V+\delta)} e^{if(X)+\omega}}{\int_{O(\delta)} e^{if(X)+\omega'}},$$

where $f(X)$ is the moment map. Localization yields the Kirillov character formula:

$$\text{tr}(e^X) = \det{}^{-\frac{1}{2}} \left(\frac{e^{\text{ad}\frac{X}{2}} - e^{-\text{ad}\frac{X}{2}}}{\text{ad}X} \right) \int_{O(\lambda_V+\delta)} e^{if(X)+\omega}.$$

5 Extensions and Applications

Woit's approach extends to:

- **Loop Groups**: The Wess-Zumino-Witten model for loop groups LG uses a similar K-theory framework, with LG/T as an infinite-dimensional Kähler manifold.
- **Chern-Simons Theory**: Witten's Chern-Simons invariants for 3-manifolds $M^3 = \Sigma \times S^1$ are analogous, with the moduli space $\mathcal{M}$ of flat connections replacing G/T. The SSQM reformulation could resolve issues in Witten's functional integral by canceling non-holomorphic sections.

6 Conclusion

Woit's SSQM path integral provides a robust alternative to coherent state path integrals, correctly quantizing the flag manifold G/T by computing the index of the Dirac operator. By incorporating fermionic variables and a damping term, it ensures only the desired representation contributes, aligning with the Borel-Weil-Bott theorem and K-theory. This approach offers a conceptual framework for quantization applicable to topological quantum theories, including loop groups and Chern-Simons theory.

References

[1] P. Woit, *Topological Quantum Theories and Representation Theory*, in *Differential Geometric Methods in Theoretical Physics*, edited by L.-L. Chau and W. Nahm, Plenum Press, New York, 1990.